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On the relation between minimizers of a Γ-limit energy and optimal lifting in BV-space

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Abstract

We study the minimizers of an energy functional which is obtained as the Γ-limit of a family of functionals depending on a small parameter $\varepsilon > 0$, associated with a function $u \in BV(\Omega, S^1)$ and a positive parameter p. We find necessary and sufficient conditions on p and the dimension under which these minimizers coincide with the optimal liftings of u, for *every* $u \in BV(\Omega, S^1)$.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u \in BV(\Omega, S^1)$, i.e., $u = (u_1, u_2) \in L^1(\Omega, \mathbb{R}^2)$, $|u(x)| = 1$ for almost every $x \in \Omega$ and the derivative of u (in the distributional sense) is a finite $2 \times N-$matrix Radon measure. The BV-seminorm of u is given by

$$\int_\Omega |Du| = \sup\left\{ \int_\Omega \sum_{k=1}^2 u_k \operatorname{div} \zeta_k \, dx \; : \; \zeta_k \in C_c^1(\Omega, \mathbb{R}^2), \; \sum_{k=1}^2 |\zeta_k(x)|^2 \le 1, \forall x \in \Omega \right\} < \infty \, ,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$. A BV lifting of u is a function $\varphi \in BV(\Omega, \mathbb{R})$ such that

$$u = e^{i\varphi} \text{ a.e. in } \Omega.$$

The existence of a BV lifting for any $u \in BV(\Omega, S^1)$ was first proved by Giaquinta, Modica and Soucek [5]. In general, we may have that

$$\min\left\{ \int_\Omega |D\varphi| \; : \; \varphi \in BV(\Omega, \mathbb{R}), \; e^{i\varphi} = u \text{ a.e. in } \Omega \right\} > \int_\Omega |Du|.$$

The optimal control of a BV lifting was given by Davila and Ignat [3] who showed the existence of a lifting $\varphi \in BV \cap L^\infty(\Omega, \mathbb{R})$ such that

$$\int_\Omega |D\varphi| \le 2 \int_\Omega |Du|. \tag{1}$$

The constant 2 in the inequality (1) is optimal for $N \ge 2$ (for example, consider

$$u(x) = \frac{x}{|x|} \tag{2}$$

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in the unit disc in $\mathbb{R}^2$, see [3] for details).

It is natural to investigate the quantity

$$E(u) = \min\left\{ \int_\Omega |D\varphi| \,:\, \varphi \in BV(\Omega,\mathbb{R}),\, e^{i\varphi} = u \text{ a.e. in } \Omega \right\}. \tag{3}$$

The case $u \in W^{1,1}$ was previously studied in [2] while the more general case $u \in BV$ was studied in [5, 7, 8]. We shall say that a lifting $\varphi \in BV(\Omega,\mathbb{R})$ of u is *optimal* if $E(u) = \int_\Omega |D\varphi|$, i.e., if φ is a minimizer in (3). An optimal lifting of u always exists but in general it is not unique (i.e., there might exist two optimal BV liftings φ_1 and φ_2 such that $\varphi_1 - \varphi_2$ is not identically constant). For example, for the function u given in (2), every optimal lifting is an argument function whose jump set is a radius of the unit disc, see [7]. The structure of an optimal lifting of u is described in [5, 8, 7] using the notion of minimal connection between singularity sets of dimension $N-2$ of u.

A natural way to approximate liftings of u is to consider, for a fixed parameter $0 < p < +\infty$, the family of energy functionals $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$ defined by

$$F_\varepsilon^{(u,p)}(\varphi) = \varepsilon \int_\Omega |\nabla\varphi|^2 + \frac{1}{\varepsilon}\int_\Omega |u - e^{i\varphi}|^p, \quad \forall \varphi \in H^1(\Omega,\mathbb{R}). \tag{4}$$

Due to the penalizing term in (4), sequences of minimizers φ_ε of $F_\varepsilon^{(u,p)}$ are expected to converge to a lifting φ_0 of u as $\varepsilon \to 0$. More precisely, Poliakovsky [9] proved that for $p > 1$ and for bounded domains Ω with Lipschitz boundary, any sequence of minimizers $\varphi_\varepsilon \in H^1(\Omega,\mathbb{R})$ of $F_\varepsilon^{(u,p)}$, satisfying $|\int_\Omega \varphi_\varepsilon| \le C$, converges strongly in L^1 (up to a subsequence) to a lifting $\varphi_0 \in BV(\Omega,\mathbb{R})$ of u as $\varepsilon \to 0$ and φ_0 is a minimizer of the $\Gamma-$limit energy $F_0^{(u,p)} : L^1(\Omega,\mathbb{R}) \to \mathbb{R}$ given by

$$F_0^{(u,p)}(\varphi) = \begin{cases} 2\displaystyle\int_{S(\varphi)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^{N-1} & \text{if } \varphi \text{ is a } BV \text{ lifting of } u, \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

Here, $S(\varphi)$ is the jump set of $\varphi \in BV(\Omega,\mathbb{R})$ and φ^-, φ^+ are the traces of φ on each of the sides of the jump set and $f^{(p)} : [0,+\infty) \to \mathbb{R}$ is the function defined by

$$f^{(p)}(\theta) = \inf_{t\in\mathbb{R}} \int_t^{\theta+t} |e^{is} - 1|^{p/2}\, ds, \; \forall \theta \ge 0.$$

Notice that $F_0^{(u,p)}(\varphi) < +\infty$ for a BV lifting φ of u since $f^{(p)}$ is an increasing Lipschitz function (see Lemma 1). Due to the fact that the energies $\{F_\varepsilon^{(u,p)}\}_{\varepsilon>0}$ and $F_0^{(u,p)}$ are invariant with respect to translations by $2\pi k$, $k \in \mathbb{Z}$, uniqueness of minimizers has a meaning up to additive constants in $2\pi\mathbb{Z}$.

The goal of this paper is to study the question whether the minimizers of $F_0^{(u,p)}$ are necessarily optimal liftings of u, for any p. Surprisingly, this turns out to be the case (in general) only in dimension one, while in dimension $N \ge 2$ this holds only for $p = 4$. Our main result is the following:

Theorem 1 *Let Ω be a bounded domain in $\mathbb{R}^N$.*

(i) If $N = 1$ then for every $u \in BV(\Omega, S^1)$ and $p \in (0,+\infty)$, φ is a minimizer of $F_0^{(u,p)}$ if and only if φ is an optimal lifting of u ;

(ii) If $N \ge 2$ then only for $p = 4$ it is true that for every $u \in BV(\Omega, S^1)$, any minimizer of $F_0^{(u,p)}$ is an optimal lifting of u.

We recall that for a function u in the smaller class $W^{1,1}(\Omega, S^1)$, a lifting of u is optimal if and only if it is a minimizer of $F_0^{(u,p)}$, for every $p \in (0, +\infty)$ (see [9]).

The paper is organized as follows. In Section 2 we recall some basic notions of BV spaces that will be needed throughout this paper. Section 3 is devoted to the one dimensional case. In Section 4 we treat the case $p = 4$, which was already studied in [9]. In Section 5 we construct counterexamples needed for the proof of assertion (ii) of Theorem 1 in the case $0 < p < 4$. For any domain Ω we construct a piecewise constant function $u \in BV(\Omega, S^1)$ depending on p such that $F_0^{(u,p)}$ has a unique minimizer ξ_0 (up to $2\pi\mathbb{Z}$ constants), u has a unique optimal lifting ζ_0 (up to $2\pi\mathbb{Z}$ constants) and $\xi_0 - \zeta_0$ is not a constant function. In Section 6, we deal with the general case $p \neq 4$. For any bounded domain G, we construct a family of functions $\{U_t\}_{t\in(-1/4,1/4)}$ that contains elements U_t with a unique optimal lifting whose energy $F_0^{(U_t,p)}$ is strictly larger than the minimal energy $\min F_0^{(U_t,p)}$. (In addition, for those functions U_t, we will prove that $F_0^{(U_t,p)}$ has a unique minimizer up to a $2\pi\mathbb{Z}$ translation.)

For the sake of simplicity of notations we shall often suppress the dependence on u and p when referring to the energies $\big\{F_\varepsilon^{(u,p)}\big\}_{\varepsilon>0}$, $F_0^{(u,p)}$ and $f^{(p)}$.

2 Preliminaries about the space BV

In this section we present some known results on BV functions that can be found in the book [1] by Ambrosio, Fusco and Pallara (see also Giusti [6] and Evans and Gariepy [4]). Let $v \in BV(\Omega, \mathbb{R}^m)$. A point $x \in \Omega$ is a point of *approximate continuity* of v if there exists $\tilde{v}(x) \in \mathbb{R}^m$ such that $\tilde{v}(x) = \underset{y\to x}{\text{ap-lim}}\, v(y)$, that is:

$$\lim_{r\to 0} \frac{\mathcal{H}^N\big(B_r(x) \cap \{y \in \Omega : \ |v(y) - \tilde{v}(x)| > \varepsilon\}\big)}{\mathcal{H}^N(B_r(x))} = 0, \quad \forall \varepsilon > 0.$$

The complement of the set of points of *approximate continuity* is denoted by $S(v)$. It is known (see [1]) that the set $S(v)$ is a countably $\mathcal{H}^{N-1}$-rectifiable Borel set, i.e., $S(v)$ is σ-finite with respect to the Hausdorff measure $\mathcal{H}^{N-1}$ and there exist countably many $N-1$ dimensional C^1-hypersurfaces $\{S_k\}_{k=1}^\infty$ such that $\mathcal{H}^{N-1}\Big(S(v) \setminus \bigcup_{k=1}^\infty S_k\Big) = 0$. Moreover, for $\mathcal{H}^{N-1}$-a.e. $x \in S(v)$ there exist $v^+(x), v^-(x) \in \mathbb{R}^m$ and a unit vector $\nu_v(x)$ such that

$$\underset{y\to x,\ \langle y-x,\nu_v(x)\rangle>0}{\text{ap-lim}} v(y) = v^+(x) \quad \text{and} \quad \underset{y\to x,\ \langle y-x,\nu_v(x)\rangle<0}{\text{ap-lim}} v(y) = v^-(x). \tag{6}$$

In the sequel we shall refer to $S(v)$ as the *jump set* of v, although (6) is valid only for $\mathcal{H}^{N-1}$-a.e. $x \in S(v)$. The vector field ν_v is called the orientation of the jump set $S(v)$. Dv is a $m \times N$ matrix valued Radon measure which can be decomposed as $Dv = D^a v + D^j v + D^c v$, where $D^a v$ is the absolutely continuous part of Dv with respect to the Lebesgue measure, while $D^j v$ and $D^c v$ are defined by

$$D^j v = Dv \llcorner S(v) \quad \text{and} \quad D^c v = (Dv - D^a v)\llcorner(\Omega \setminus S(v)).$$

We shall call $D^j v$ and $D^c v$ the jump part and the Cantor part, respectively, of Dv. We have:

1. $D^a v = \nabla v\, \mathcal{H}^N$ where $\nabla v \in L^1(\Omega, \mathbb{R}^{m\times N})$ is the approximate differential of v;
2. $(D^c v)(B) = 0$ for any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{N-1}$;
3. $D^j v = (v^+ - v^-) \otimes \nu_v\, \mathcal{H}^{N-1} \llcorner S(v)$.

Throughout this paper we identify the function v with its precise representative $v^* : \Omega \mapsto \mathbb{R}^m$ given by

$$v^*(x) = \lim_{r\to 0} \frac{1}{\mathcal{H}^N(B_r(x))} \int_{B_r(x)} v(y)\,dy\,,$$

if this limit exists, and $v^*(x) = 0$ otherwise. Note that v^* specifies the values of v except on a $\mathcal{H}^{N-1}$-negligible set.

We also recall Vol'pert's chain rule. Let Ω be a bounded domain and assume that $v \in BV(\Omega, \mathbb{R}^m)$ and $g \in [C^1(\mathbb{R}^m)]^q$ is a Lipschitz function. Then $w = g \circ v$ belongs to $BV(\Omega, \mathbb{R}^q)$ and

$$D^a w = \nabla g(v)\,\nabla v\,\mathcal{H}^N,\; D^c w = \nabla g(v)\,D^c v,\; D^j w = \big[g(v^+) - g(v^-)\big] \otimes \nu_v\,\mathcal{H}^{N-1} \llcorner S(v)\,. \tag{7}$$

3 The one-dimensional case

In this section we shall show that the optimal liftings of u coincide with the minimizers of $F_0^{(u,p)}$ in the one-dimensional case, for every parameter $p > 0$ and any function $u \in BV(\Omega, S^1)$. The proof uses the same method as in [8].

Proof of *(i)* in Theorem 1. Let Ω be an interval in $\mathbb{R}$ and let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. By the chain rule (7), it follows that

$$(\dot\varphi)^a + (\dot\varphi)^c = u \wedge ((\dot u)^a + (\dot u)^c) \;\text{ and }\; (\dot\varphi)^j = \sum_{a\in S(u)} (\varphi(a+) - \varphi(a-))\delta_a + \sum_{b\in B} (\varphi(b+) - \varphi(b-))\delta_b \tag{8}$$

where $B \subset \Omega$ is a finite set such that $S(u) \cap B = \emptyset$ and $\varphi(b+) - \varphi(b-) = -2\pi\alpha_b$, $\alpha_b \in \mathbb{Z}$, for every $b \in B$. For any $a \in S(u)$, we denote $d_a(u) = \operatorname{Arg} \dfrac{u(a+)}{u(a-)}$ where $\operatorname{Arg} \omega \in (-\pi, \pi]$ is the argument of the unit complex number ω. Since $f^{(p)}$ is increasing and $|\varphi(a+) - \varphi(a-)| \geq |d_a(u)|$ in $S(u)$, it follows that

$$f^{(p)}(|\varphi(a+) - \varphi(a-)|) \geq f^{(p)}(|d_a(u)|) \text{ if } a \in S(u) \;\text{ and }\; f^{(p)}(|\varphi(b+) - \varphi(b-)|) \geq 0 \text{ if } b \in B \tag{9}$$

with equality if and only if

$$|\varphi(a+) - \varphi(a-)| = |d_a(u)| \text{ for } a \in S(u) \quad\text{and}\quad \alpha_b = 0 \text{ for } b \in B. \tag{10}$$

According to (8), we have

$$\int_\Omega \Big(|(\dot\varphi)^a| + |(\dot\varphi)^c|\Big) = \int_\Omega \Big(|(\dot u)^a| + |(\dot u)^c|\Big).$$

By [8], it follows that

$$E(u) = \int_\Omega \Big(|(\dot u)^a| + |(\dot u)^c|\Big) + \sum_{a\in S(u)} |d_a(u)|,$$

i.e., φ is an optimal lifting if $\displaystyle\int_\Omega |(\dot\varphi)^j| = \sum_{a\in S(u)} |d_a(u)|$. Therefore, by (9) and (10), we obtain that

$$\min F_0^{(u,p)} = 2 \sum_{a\in S(u)} f^{(p)}(|d_a(u)|).$$

Finally, we conclude that φ is a minimizer of $F_0^{(u,p)}$ if and only if φ is an optimal lifting of u. $\square$

4 The case $p = 4$

In this section we shall recall the proof from [9] of the result that states that for $p = 4$ minimizers of the Γ-limit energy $F_0^{(u,p)}$ coincide with those of the energy $E(u)$ in (3) for every $u \in BV(\Omega, S^1)$. We also derive an asymptotic upper bound for the minimal energy of $F_\varepsilon^{(u,4)}$ in terms of the mass of the measure $|Du|$.

Proof of *(ii)* of Theorem 1 for $p = 4$. Let $\varphi \in BV(\Omega, \mathbb{R})$ be a lifting of u. Then $|u^+ - u^-| = 2\big|\sin\dfrac{\varphi^+ - \varphi^-}{2}\big|$ $\mathcal{H}^{N-1}$-a.e. in $S(u)$. A simple computation yields

$$f^{(4)}(\theta) = 2\theta - 4\big|\sin\frac{\theta}{2}\big|, \ \forall\theta \geq 0.$$

This implies that

$$F_0^{(u,4)}(\varphi) = 4\int_{S(\varphi)} |\varphi^+ - \varphi^-|\, d\mathcal{H}^{N-1} - 4\int_{S(u)} |u^+ - u^-|\, d\mathcal{H}^{N-1}.$$

On the other hand, the chain rule (7) yields that

$$D^a\varphi = u \wedge D^a u \quad \text{and} \quad D^c\varphi = u \wedge D^c u \tag{11}$$

and therefore, the total variation of the diffuse part of $D\varphi$ is completely determined by Du, i.e.,

$$\int_\Omega (|D^a\varphi| + |D^c\varphi|) = \int_\Omega (|D^a u| + |D^c u|). \tag{12}$$

Hence, φ is a minimizer of $F_0^{(u,4)}$ if and only if φ is an optimal lifting of u. □

As a consequence, we deduce an estimate for the energy $F_\varepsilon^{(u,4)}$ which relies on some results from [3] and [9].

Corollary 1 *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $u \in BV(\Omega, S^1)$. Then*

$$\min F_\varepsilon^{(u,4)} \leq 4\int_\Omega |Du| + o(1)$$

where $o(1)$ is a quantity that tends to 0 as $\varepsilon \to 0$.

Proof. By contradiction, assume that there exist a constant $\delta > 0$ and a sequence $\{\varepsilon_k\}_{k\geq 1}$ tending to 0 as $k \to \infty$, such that

$$F_{\varepsilon_k}^{(u,4)}(\varphi_{\varepsilon_k}) \geq 4\int_\Omega |Du| + \delta\,, \tag{13}$$

where $\varphi_{\varepsilon_k} \in H^1(\Omega, \mathbb{R})$ is a minimizer of $F_{\varepsilon_k}^{(u,4)}$. Since the value of $F_{\varepsilon_k}^{(u,4)}(\varphi_{\varepsilon_k})$ does not change by adding a constant multiple of 2π to φ_{ε_k}, we may assume that $0 \leq \int_\Omega \varphi_{\varepsilon_k}\, dx \leq 2\pi\mathcal{H}^N(\Omega)$. According to [9] it follows that, up to a subsequence,

$$\varphi_{\varepsilon_k} \to \varphi_0 \ \text{ in } \ L^1 \quad \text{and} \quad \lim_{k\to\infty} F_{\varepsilon_k}^{(u,4)}(\varphi_{\varepsilon_k}) = F_0^{(u,4)}(\varphi_0)\,,$$

where φ_0 is a BV lifting of u that minimizes the $\Gamma-$limit energy $F_0^{(u,4)}$. Using (13), it follows that

$$F_0^{(u,4)}(\varphi_0) \geq 4\int_\Omega |Du| + \delta. \tag{14}$$

On the other hand, by assertion *(ii)* of Theorem 1 in the case $p = 4$, we know that φ_0 is an optimal lifting and

$$F_0^{(u,4)}(\varphi_0) = 4\int_{S(\varphi_0)} |\varphi_0^+ - \varphi_0^-|\, d\mathcal{H}^{N-1} - 4\int_{S(u)} |u^+ - u^-|\, d\mathcal{H}^{N-1}.$$

By (1) we deduce that $\int_\Omega |D\varphi_0| \le 2\int_\Omega |Du|$ and therefore, it implies by (12),

$$F_0^{(u,4)}(\varphi_0) \le 4\int_\Omega |Du|$$

which contradicts (14). $\square$

It would be interesting to have a direct proof of Corollary 1 which does not use the results in [3] and [9]. That will lead to a new proof of the inequality (1).

5 The case $p \in (0, 4)$

In this section we prove the case $p < 4$ of assertion *(ii)* of Theorem 1. We shall first construct, for each $0 < p < 4$, a piecewise constant function $u \in BV(\mathcal{R}, S^1)$ in a rectangle $\mathcal{R} \subset \mathbb{R}^2$ such that no minimizer of $F_0^{(u,p)}$ is an optimal lifting of u. Then, we shall adapt this example to the case of an arbitrary bounded domain Ω.

We start by two preliminary results about the function $f^{(p)}$:

Lemma 1 *Let $0 < p < \infty$. The function $f^{(p)}$ is an increasing Lipschitz continuous function. Moreover,*

$$f^{(p)}(\theta) = \begin{cases} \displaystyle\int_{-\theta/2}^{\theta/2} |e^{is} - 1|^{p/2}\, ds & \text{if } \theta \in [2\pi k, 2\pi(k+1)],\ k \text{ even}, \\ \displaystyle\int_{-\theta/2+\pi}^{\theta/2+\pi} |e^{is} - 1|^{p/2}\, ds & \text{if } \theta \in [2\pi k, 2\pi(k+1)],\ k \text{ odd}. \end{cases} \tag{15}$$

Proof. In the sequel we shall write for short f instead of $f^{(p)}$. The function

$$s \in \mathbb{R} \mapsto |e^{is} - 1|^{p/2} = 2^{p/2}\left|\sin\frac{s}{2}\right|^{p/2}$$

is 2π-periodic, increasing on $(0, \pi)$ and symmetric with respect to π. Hence, if $\theta \in [0, 2\pi]$, then $f(\theta) = \displaystyle\int_{-\theta/2}^{\theta/2} |e^{is} - 1|^{p/2}\, ds$. In general, if $\theta = 2\pi k + \tilde{\theta}$ with $\tilde{\theta} \in [0, 2\pi]$ and $k \in \mathbb{N}$, we have $f(\theta) = f(2\pi k) + f(\tilde{\theta})$ and (15) is now straightforward. In particular, we deduce that

$$f(2\pi k) = k f(2\pi), \quad \forall k \in \mathbb{N}. \tag{16}$$

From here, we conclude that almost everywhere in $(0, +\infty)$, f is differentiable and $0 < f' \le 2^{p/2}$. $\square$

Lemma 2 *Let $0 < p < 4$. Then the function $\theta \in (0, \pi) \mapsto \dfrac{f^{(p)}(2\pi - \theta) - f^{(p)}(\theta)}{\pi - \theta}$ is increasing.*

Proof. It is sufficient to prove that the function $g : (0, \pi) \to \mathbb{R}$ defined by

$$g(\theta) = f(2\pi - \theta) - f(\theta) - (\pi - \theta)\Big(f'(2\pi - \theta) + f'(\theta)\Big)$$

is positive, where we denoted $f = f^{(p)}$ as above. Indeed, by Lemma 1 we have for every $\theta \in (0, \pi)$,

$$g'(\theta) = (\pi - \theta)\big(f''(2\pi - \theta) - f''(\theta)\big) = p\, 2^{p/2-4}\, (\pi - \theta)\, \sin\frac{\theta}{2}\Big(\cos^{p/2-2}\frac{\theta}{4} - \sin^{p/2-2}\frac{\theta}{4}\Big).$$

Since $p < 4$ it follows that $g'(\theta) < 0$, $\forall\theta \in (0, \pi)$; hence g is decreasing. Since $\lim_{\theta\to\pi} g(\theta) = 0$, we deduce that g must be positive on $(0, \pi)$. $\square$

Construction of a counter-example u when Ω is a rectangle. Let $p \in (0, 4)$. We first construct our function u in a certain rectangle $\mathcal{R}$. Let $\theta_1 = \frac{4\pi}{5}$ and $\theta_2 = \frac{3\pi}{4}$. Thanks to Lemma 2 we can choose $L_3 > L_1 > 0$ such that

$$\frac{5}{4} = \frac{\pi - \theta_2}{\pi - \theta_1} > \frac{L_3}{L_1} > \frac{f^{(p)}(2\pi - \theta_2) - f^{(p)}(\theta_2)}{f^{(p)}(2\pi - \theta_1) - f^{(p)}(\theta_1)} > 1. \tag{17}$$

Set also $L_2 = L_3$ and $L_4 = L_3$. We consider the rectangle

$$\mathcal{R} = \Big\{(x, y) \in \mathbb{R}^2 \,:\, -L_2 < x < L_4,\ -L_3 < y < L_1\Big\}.$$

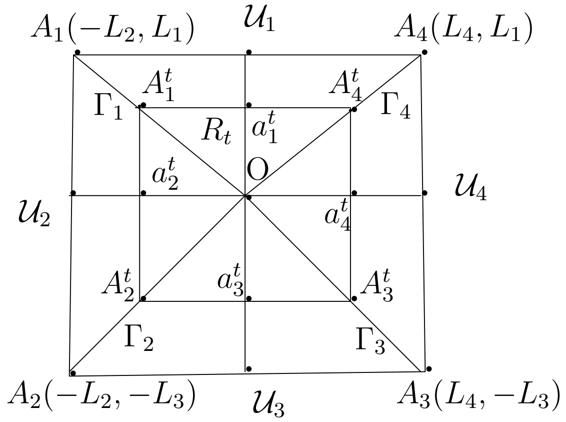

Figure 1: The rectangle construction for $p \in (0, 4)$

Notice that the rectangle $\mathcal{R}$ depends on p by the choice of the edges; moreover, the choice (17) is no longer possible for $p \geq 4$. In the rectangle $\mathcal{R}$, we denote the vertices $A_1 = (-L_2, L_1)$, $A_2 = (-L_2, -L_3)$, $A_3 = (L_4, -L_3)$ and $A_4 = (L_4, L_1)$ and also the interior full triangles $\mathcal{U}_k = \triangle\, A_k O A_{k-1}$ and the segments $\Gamma_k = (OA_k)$ for $1 \leq k \leq 4$ where $O = (0, 0)$ is the origin and we use the convention that $A_0 = A_4$, see Figure 1.

Let $\varphi_0 \in BV(\mathcal{R}, \mathbb{R})$ be the piecewise constant function defined by

$$\varphi_0(x,y) = \begin{cases} \frac{\pi}{2} & \text{if} \quad 0 < x < L_4, \quad 0 < y < L_1, \\ \frac{5\pi}{4} & \text{if} \quad -L_2 < x < 0, \quad 0 < y < L_1, \\ \frac{3\pi}{2} & \text{if} \quad -L_2 < x < 0, \quad -L_3 < y < 0, \\ \frac{3\pi}{10} & \text{if} \quad 0 < x < L_4, \quad -L_3 < y < 0 \end{cases}$$

and set $u = e^{i\varphi_0} \in BV(\mathcal{R}, S^1)$.

In Lemmas 3 and 4 below we shall prove that φ_0 is the unique optimal lifting of u (up to a $2\pi\mathbb{Z}$ constant) and φ_0 is not a minimizer of $F_0^{(u,p)}$. Actually, we prove that the lifting $\psi_0 \in BV(\mathcal{R}, \mathbb{R})$ of u defined as

$$\psi_0(x,y) = \begin{cases} \frac{\pi}{2} & \text{if} \quad 0 < x < L_4, \quad 0 < y < L_1, \\ -\frac{3\pi}{4} & \text{if} \quad -L_2 < x < 0, \quad 0 < y < L_1, \\ -\frac{\pi}{2} & \text{if} \quad -L_2 < x < 0, \quad -L_3 < y < 0, \\ \frac{3\pi}{10} & \text{if} \quad 0 < x < L_4, \quad -L_3 < y < 0 \end{cases}$$

is the unique minimizer of $F_0^{(u,p)}$ (up to $2\pi\mathbb{Z}$ constants).

Lemma 3 *The function φ_0 is the unique optimal lifting of u (up to a $2\pi\mathbb{Z}$ constant).*

Proof. Let $\varphi \in BV(\mathcal{R}, \mathbb{R})$ be a lifting of u. Then

$$\int_{\mathcal{R}} |D\varphi| = \sum_{k=1}^{4} \left(\int_{\mathcal{U}_k} |D\varphi| + \int_{\Gamma_k} |\varphi_{\Gamma_k}^+ - \varphi_{\Gamma_k}^-| \, d\mathcal{H}^1 \right)$$

where $\varphi_{\Gamma_k}^+$ and $\varphi_{\Gamma_k}^-$ are the traces of φ on Γ_k. Let us consider the one-dimensional sections

$$\mathcal{R}_t = \Big\{ (tx, ty) \, : \, (x,y) \in \partial\mathcal{R} \Big\}, \ \forall t \in (0,1)$$

where we denote the vertices of the rectangle $\mathcal{R}_t$ by $\{A_k^t\}_{1 \le k \le 4}$. By the characterization of BV functions by sections (see Theorem 3.103 in [1]), the restriction $\varphi_t = \varphi\big|_{\mathcal{R}_t}$ belongs to $BV(\mathcal{R}_t, \mathbb{R})$ for almost any $t \in (0,1)$. We define the following rescaled variation of φ_t on $\mathcal{R}_t$ as

$$V(\varphi_t, \mathcal{R}_t) = \sum_{k=1}^{4} \left(L_k \int_{\mathcal{R}_t \cap \mathcal{U}_k} \Big| \frac{\partial \varphi_t}{\partial \tau} \Big| + \sqrt{L_k^2 + L_{k+1}^2} \, \big| \varphi_{\Gamma_k}^+(A_k^t) - \varphi_{\Gamma_k}^-(A_k^t) \big| \right) \quad \text{for a.e. } t \in (0,1)$$

so that

$$\int_0^1 V(\varphi_t, \mathcal{R}_t) \, dt \le \int_{\mathcal{R}} |D\varphi|$$

(here τ is the tangent vector of straight lines). An easy computation yields

$$\int_{\mathcal{R}} |D\varphi_0| = L_1 \frac{3\pi}{4} + L_2 \frac{\pi}{4} + L_3 \frac{6\pi}{5} + L_4 \frac{\pi}{5}.$$

In order to prove that φ_0 is an optimal lifting, it is sufficient to prove that

$$V(\varphi_t, \mathcal{R}_t) \ge L_1 \frac{3\pi}{4} + L_2 \frac{\pi}{4} + L_3 \frac{6\pi}{5} + L_4 \frac{\pi}{5} \quad \text{for a.e. } t \in (0,1). \tag{18}$$

We shall use a method from [8]. Denoting the restriction of u to $\mathcal{R}_t$ by $u_t = u\big|_{\mathcal{R}_t}$, we have for almost every $t \in (0,1)$: $u_t = e^{i\varphi_t}$ $\mathcal{H}^1$ − a.e. in $\mathcal{R}_t$ and $S(u_t) = \{a_k^t : 1 \le k \le 4\}$ where $a_k^t = \mathcal{R}_t \cap \mathcal{U}_k \cap \{x = 0\}$ for $k \in \{1,3\}$ and $a_k^t = \mathcal{R}_t \cap \mathcal{U}_k \cap \{y = 0\}$ for $k \in \{2,4\}$. The chain rule (7) leads to

$$\left(\frac{\partial \varphi_t}{\partial \tau}\right)^a = u_t \wedge \left(\frac{\partial u_t}{\partial \tau}\right)^a = 0 \quad \text{and} \quad \left(\frac{\partial \varphi_t}{\partial \tau}\right)^c = u_t \wedge \left(\frac{\partial u_t}{\partial \tau}\right)^c = 0;$$

hence,

$$\frac{\partial \varphi_t}{\partial \tau} = \left(\frac{\partial \varphi_t}{\partial \tau}\right)^j = \sum_{a \in S(u_t)} (\varphi_t(a+) - \varphi_t(a-))\delta_a + \sum_{b \in \mathcal{B}} (\varphi_t(b+) - \varphi_t(b-))\delta_b.$$

Here, the Lipschitz curve $\mathcal{R}_t$ is considered oriented counterclockwise and the traces of φ_t are taken with respect to this orientation. We have that

1. $\mathcal{B} \subset \mathcal{R}_t$ is a finite set such that $S(u_t) \cap \mathcal{B} = \emptyset$ and $\varphi_t(b+) - \varphi_t(b-) = -2\pi\alpha_b$ where $\alpha_b \in \mathbb{Z}, \forall b \in \mathcal{B}$;
2. $\varphi_t(a+) - \varphi_t(a-) = \operatorname{Arg} \frac{u_t(a+)}{u_t(a-)} - 2\pi\alpha_a$ with $\alpha_a \in \mathbb{Z}, \forall a \in S(u_t)$.

Therefore, setting $L_5 = L_1$, it follows that

$$V(\varphi_t, \mathcal{R}_t) = \sum_{k=1}^{4} \left(\sum_{a \in (S(u_t) \cup \mathcal{B}) \cap \mathcal{U}_k} L_k \big|\varphi_t(a+) - \varphi_t(a-)\big| + \sqrt{L_k^2 + L_{k+1}^2}\, \big|\varphi_{\Gamma_k}^+(A_k^t) - \varphi_{\Gamma_k}^-(A_k^t)\big| \right). \tag{19}$$

Since $\displaystyle\int_{\mathcal{R}_t} \frac{\partial \varphi_t}{\partial \tau} = 0$, we get

$$\sum_{a \in S(u_t) \cup \mathcal{B}} \alpha_a = \frac{1}{2\pi} \sum_{a \in S(u_t)} \operatorname{Arg} \frac{u_t(a+)}{u_t(a-)} = 1. \tag{20}$$

Obviously,

$$|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)| \ge \Big| \operatorname{Arg} \frac{u_t(a_k^t+)}{u_t(a_k^t-)} \Big|, \ \forall 1 \le k \le 4.$$

By (19), the inequality (18) will follow from the surplus of the variation induced by the condition (20), i.e.,

$$V(\varphi_t, \mathcal{R}_t) \ge L_3 \frac{2\pi}{5} + \sum_{k=1}^{4} L_k \Big| \operatorname{Arg} \frac{u_t(a_k^t+)}{u_t(a_k^t-)} \Big|. \tag{21}$$

Indeed, suppose that there is $b \in \mathcal{B}$ such that $\alpha_b \ne 0$. If $b \in \mathcal{U}_k$ for some $1 \le k \le 4$ then by (17),

$$L_k |\varphi_t(b+) - \varphi_t(b-)| \ge 2\pi L_k > L_3 \frac{2\pi}{5}.$$

If $b = A_k^t$ for some $1 \le k \le 4$, then

$$\sqrt{L_k^2 + L_{k+1}^2}\, |\varphi_{\Gamma_k}^+(A_k^t) - \varphi_{\Gamma_k}^-(A_k^t)| \ge 2\pi \sqrt{L_k^2 + L_{k+1}^2} > L_3 \frac{2\pi}{5}$$

(here we used the fact that the traces of φ_t on Γ_k coincide with $\varphi_{\Gamma_k}^{\pm}(A_k^t)$ for a.e. $t \in (0,1)$). Otherwise, according to (20), there exists $\alpha_a \ne 0$ for some $a = a_k^t$ and by (17), we easily check that

$$L_k |\varphi_t(a_k^t+) - \varphi_t(a_k^t-)| \ge L_3 \frac{2\pi}{5} + L_k \Big| \operatorname{Arg} \frac{u_t(a_k^t+)}{u_t(a_k^t-)} \Big|$$

with equality if and only if $k = 3$. Therefore, (21) holds, i.e., φ_0 is an optimal lifting of u.

It remains to prove the uniqueness of the optimal lifting φ_0 (up to a $2\pi\mathbb{Z}$ constant). Let φ be an optimal lifting. From above, we deduce that the restriction φ_t on $\mathcal{R}_t$ satisfies for almost $t \in (0,1)$ that

$$S(\varphi_t) = S(u_t) \quad \text{and} \quad \alpha_{a_k^t} = \begin{cases} 0 & \text{if } k \in \{1,2,4\}, \\ 1 & \text{if } k = 3. \end{cases} \tag{22}$$

It follows that

$$\int_{\mathcal{R}} |D\varphi| \geq \int_{S(\varphi)} |\varphi^+ - \varphi^-| \, d\mathcal{H}^1 \geq \int_{S(u)} |\varphi^+ - \varphi^-| \, d\mathcal{H}^1$$
$$\geq \int_0^1 \sum_{k=1}^4 L_k |\varphi_t(a_k^t+) - \varphi_t(a_k^t-)| \, dt = \int_{\mathcal{R}} |D\varphi_0|.$$

Since φ is an optimal lifting, we deduce that $S(\varphi) = S(u)$. By (11), we have $D^a\varphi = D^c\varphi = 0$. It follows that φ is constant on each connected component of $\mathcal{R} \setminus S(u)$. By (22), we conclude that $\varphi - \varphi_0$ is a constant function, for some constant in $2\pi\mathbb{Z}$. $\square$

Lemma 4 *The function ψ_0 is the unique minimizer of $F_0^{(u,p)}$ (up to $2\pi\mathbb{Z}$ constants).*

Proof. We use the same argument and notations as in the proof of Lemma 3. Let $\varphi \in BV(\mathcal{R}, \mathbb{R})$ be a lifting of u. By (11), we have $D^a\varphi = D^c\varphi = 0$ and $D\varphi = D^j\varphi = (\varphi^+ - \varphi^-)\nu_\varphi \mathcal{H}^1 \llcorner S(\varphi)$. We define for almost every $t \in (0,1)$ the following variation of φ_t on $\mathcal{R}_t$:

$$G(\varphi_t, \mathcal{R}_t) = \sum_{k=1}^4 \Bigg(\sum_{a \in (S(u_t) \cup \mathcal{B}) \cap \mathcal{U}_k} L_k f^{(p)}\big(|\varphi_t(a+) - \varphi_t(a-)|\big)$$
$$+ \sqrt{L_k^2 + L_{k+1}^2} \, f^{(p)}\big(|\varphi_{\Gamma_k}^+(A_k^t) - \varphi_{\Gamma_k}^-(A_k^t)|\big) \Bigg)$$

so that

$$2\int_0^1 G(\varphi_t, \mathcal{R}_t) \, dt \leq F_0^{(u,p)}(\varphi).$$

In order to prove that ψ_0 is a minimizer of $F_0^{(u,p)}$, it is sufficient to verify that

$$G(\varphi_t, \mathcal{R}_t) \geq L_1 f^{(p)}(\frac{5\pi}{4}) + L_2 f^{(p)}(\frac{\pi}{4}) + L_3 f^{(p)}(\frac{4\pi}{5}) + L_4 f^{(p)}(\frac{\pi}{5}) = \frac{F_0^{(u,p)}(\psi_0)}{2} \quad \text{for a.e. } t \in (0,1). \tag{23}$$

Indeed, suppose that there is $b \in \mathcal{B}$ such that $\alpha_b \neq 0$. If $b \in \mathcal{U}_k$ for some $1 \leq k \leq 4$ then by (17) and Lemma 1,

$$L_k f^{(p)}(|\varphi_t(b+) - \varphi_t(b-)|) + L_1 f^{(p)}(|\varphi_t(a_1^t+) - \varphi_t(a_1^t-)|) > L_1 f^{(p)}(\frac{5\pi}{4})$$

and then, we use that

$$f^{(p)}(|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)|) \geq f^{(p)}\Bigg(\Big|\operatorname{Arg} \frac{u_t(a_k^t+)}{u_t(a_k^t-)}\Big|\Bigg), \ 2 \leq k \leq 4.$$

If $b = A_k^t$ for some $1 \leq k \leq 4$, then

$$\sqrt{L_k^2 + L_{k+1}^2} \, f^{(p)}(|\varphi_{\Gamma_k}^+(A_k^t) - \varphi_{\Gamma_k}^-(A_k^t)|) + L_1 f^{(p)}(|\varphi_t(a_1^t+) - \varphi_t(a_1^t-)|) > L_1 f^{(p)}(\frac{5\pi}{4}).$$

Otherwise, according to (20), there exists $\alpha_a \neq 0$ for some $a = a_k^t$. By Lemma 1, we notice that the map $\theta \in (0,\pi) \mapsto f^{(p)}(2\pi - \theta) - f^{(p)}(\theta)$ is decreasing. Then, by (17), we easily check that

$$L_k f^{(p)}(|\varphi_t(a_k^t+) - \varphi_t(a_k^t-)|) + L_1 f^{(p)}\Big(\big|\operatorname{Arg}\frac{u_t(a_1^t+)}{u_t(a_1^t-)}\big|\Big) \geq L_k f^{(p)}\Big(\big|\operatorname{Arg}\frac{u_t(a_k^t+)}{u_t(a_k^t-)}\big|\Big) + L_1 f^{(p)}(\frac{5\pi}{4})$$

with equality if and only if $k = 1$. Therefore, (23) holds and we also deduce that if φ is a minimizer of $F_0^{(u,p)}$, then for almost every $t \in (0,1)$,

$$S(\varphi_t) = S(u_t) \quad \text{and} \quad \alpha_{a_k^t} = \begin{cases} 0 & \text{if } 2 \leq k \leq 4, \\ 1 & \text{if } k = 1. \end{cases} \tag{24}$$

The uniqueness of the minimizer ψ_0 (up to $2\pi\mathbb{Z}$ constants) follows by (24) as in the proof of Lemma 3. $\square$

Proof of *(ii)* in Theorem 1 for $p \in (0,4)$. Let Ω be an arbitrary bounded domain in $\mathbb{R}^N$, for $N \geq 2$. Denote by $\mathcal{D} = (2\mathcal{R}) \times (-2,2)^{N-2} \subset \mathbb{R}^N$. By translating and shrinking homotopically the rectangular parallelepiped $\mathcal{D}$, we may suppose that $\mathcal{D} \subset\subset \Omega$. Let u, φ_0 and ψ_0 be the functions in $\mathcal{R}$ constructed above and denote $\mathcal{D}_1 = \mathcal{R} \times (-1,1)^{N-2}$. We write $x = (x_1, x_2, \ldots, x_N) = (x_1, x_2, x') \in \mathbb{R}^N$. We define in Ω,

$$w(x) = \begin{cases} u(x_1, x_2) & \text{in } \mathcal{D}_1, \\ 1 & \text{in } \big(\mathcal{D} \setminus \mathcal{D}_1\big) \cap \{x_1 > 0\}, \\ -1 & \text{otherwise.} \end{cases}$$

Consider the liftings

$$\zeta_0(x) = \begin{cases} \varphi_0(x_1, x_2) & \text{in } \mathcal{D}_1, \\ 0 & \text{in } \big(\mathcal{D} \setminus \mathcal{D}_1\big) \cap \{x_1 > 0\}, \\ \pi & \text{otherwise} \end{cases}$$

and

$$\xi_0(x) = \begin{cases} \psi_0(x_1, x_2) & \text{in } \mathcal{D}_1, \\ 0 & \text{in } \big(\mathcal{D} \setminus \mathcal{D}_1\big) \cap \{x_1 > 0\}, \\ -\pi & \text{otherwise.} \end{cases}$$

We prove that ζ_0 is the unique optimal lifting of w and ξ_0 is the unique minimizer of $F_0^{(w,p)}$, but $\zeta_0 - \xi_0$ is not constant since

$$\zeta_0 = \begin{cases} \xi_0 & \text{in } \mathcal{D} \cap \{x_1 > 0\}, \\ \xi_0 + 2\pi & \text{otherwise.} \end{cases}$$

Step 1. The function ζ_0 is the unique optimal lifting of w (up to a $2\pi\mathbb{Z}$ constant).

Indeed, let $\zeta \in BV(\Omega, \mathbb{R})$ be a lifting of w. Obviously, $|\zeta^+ - \zeta^-| \geq d_{S^1}(w^+, w^-) = |\zeta_0^+ - \zeta_0^-|$ $\mathcal{H}^{N-1}$-a.e. in $S(w) \cap \big(\Omega \setminus \mathcal{D}_1\big)$. The restriction of ζ to $\mathcal{R} \times \{x'\}$ is a BV lifting of u for almost every $x' \in (-1,1)^{N-2}$. Therefore, by Lemma 3, we obtain

$$\begin{aligned} \int_\Omega |D\zeta| &= \int_{\Omega \setminus \mathcal{D}_1} |D\zeta| + \int_{\mathcal{D}_1} |D\zeta| \\ &\geq \int_{S(w) \cap (\Omega \setminus \mathcal{D}_1)} |\zeta^+ - \zeta^-| \, d\mathcal{H}^{N-1} + \int_{(-1,1)^{N-2}} dx' \int_{\mathcal{R} \times \{x'\}} \Big|\big(\frac{\partial \zeta}{\partial x_1}, \frac{\partial \zeta}{\partial x_2}\big)\Big| \\ &\geq \int_{S(w) \cap (\Omega \setminus \mathcal{D}_1)} d_{S^1}(w^+, w^-) \, d\mathcal{H}^{N-1} + 2^{N-2} \int_{\mathcal{R}} |D\varphi_0| = \int_\Omega |D\zeta_0|, \end{aligned}$$

i.e., ζ_0 is an optimal lifting of w. Let now ζ be an optimal lifting. From the above it follows that

$$\int_{\Omega\setminus\mathcal{D}_1} |D\zeta| = \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} d_{S^1}(w^+,w^-)\,d\mathcal{H}^{N-1}$$

and for almost every $x' \in (-1,1)^{N-2}$, the restriction of ζ to $\mathcal{R}\times\{x'\}$ is an optimal lifting of u, i.e.,

$$\int_{\mathcal{R}\times\{x'\}} |D\zeta| = \int_{\mathcal{R}} |D\varphi_0|.$$

As in the proof of Lemma 3, it follows that $\zeta - \zeta_0 \equiv 2\pi m$ in $\mathcal{D}_1$ where $m \in \mathbb{Z}$. Since the size of the jump of ζ must satisfy $0 < d_{S^1}(w^+,w^-) < \pi$ on $\partial\mathcal{D}$, we deduce that

$$\zeta - \zeta_0 \equiv 2\pi m \quad \text{in} \quad \Omega.$$

Hence, ζ_0 is the unique optimal lifting of w (up to $2\pi\mathbb{Z}$ constants).

Step 2. The function ξ_0 is the unique minimizer of $F_0^{(w,p)}$ (up to $2\pi\mathbb{Z}$ constants).

As in *Step 1*, using Lemma 4, we have that for every BV lifting ζ of w,

$$\begin{aligned}
\frac{F_0^{(w,p)}(\zeta)}{2} &= \int_{S(\zeta)\cap(\Omega\setminus\mathcal{D}_1)} f^{(p)}(|\zeta^+-\zeta^-|)\,d\mathcal{H}^{N-1} + \int_{S(\zeta)\cap\mathcal{D}_1} f^{(p)}(|\zeta^+-\zeta^-|)\,d\mathcal{H}^{N-1} \\
&\geq \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} f^{(p)}(|\zeta^+-\zeta^-|)\,d\mathcal{H}^{N-1} \\
&\quad + \int_{(-1,1)^{N-2}} dx' \int_{S(\zeta)\cap(\mathcal{R}\times\{x'\})} f^{(p)}(|\zeta^+-\zeta^-|)\,d\mathcal{H}^1 \\
&\geq \int_{S(w)\cap(\Omega\setminus\mathcal{D}_1)} f^{(p)}\big(d_{S^1}(w^+,w^-)\big)\,d\mathcal{H}^{N-1} + 2^{N-3}F_0^{(u,p)}(\psi_0) = \frac{F_0^{(w,p)}(\xi_0)}{2}
\end{aligned}$$

i.e., ξ_0 is a minimizer of $F_0^{(w,p)}$. The uniqueness of the minimizer follows by the same argument as above. $\square$

6 Proof of *(ii)* in Theorem 1 for $p \neq 4$

In this section we shall complete the proof of our main result in the general case $p \in (0,4)\cup(4,+\infty)$. The strategy will be to construct a family of functions $\mathcal{U} = \{U_t\}_{t\in(-\frac14,\frac14)}$ in $BV(\Omega,S^1)$ with the following property: for every $p \neq 4$, there exists a function U_t in the family $\mathcal{U}$ such that U_t has a unique optimal lifting (up to translations in $2\pi\mathbb{Z}$) and the energy $F_0^{(U_t,p)}$ of the optimal lifting is larger than the minimal energy $\min F_0^{(U_t,p)}$. First of all, we make that construction in the special case of the two-dimensional disc

$$\Omega := \{z \in \mathbb{C} : |z| < 2\}.$$

Construction of the family $\mathcal{U} = \{U_t\}_{t\in(-\frac14,\frac14)}$ in the disc $\Omega = B(0,2) \subset \mathbb{R}^2$. For any $z \in \Omega\setminus\{0\}$, we denote the argument $\bar\theta(z) \in [0,2\pi)$, i.e., $\frac{z}{|z|} = e^{i\bar\theta(z)}$. Let $t \in (-\frac14,\frac14)$. We define the set

$$A_t := \{\, z \in \Omega : \ z = re^{i\theta},\ r \in (1,2),\ 0 < \theta < (\frac34 + t)\ln r \,\}$$

and we consider the function $\hat\theta_t : \Omega \to \mathbb{R}$ given by

$$\hat\theta_t(z) := \bar\theta(z) + 2\pi\chi_{A_t}(z), \quad \forall z \in \Omega, \tag{25}$$

where χ_{A_t} is the characteristic function associated to the set A_t. Now let $U_t \in BV(\Omega, S^1)$ be defined by

$$U_t(z) := e^{i\frac{9}{10}\hat{\theta}_t(z)}, \quad \forall z \in \Omega. \tag{26}$$

Set the liftings $\varphi_{1,t}, \varphi_{2,t} \in BV(\Omega, \mathbb{R})$ of U_t:

$$\varphi_{1,t} := \frac{9}{10}\hat{\theta}_t = \frac{9}{10}\bar{\theta} + \frac{9\pi}{5}\chi_{A_t} \quad \text{and} \quad \varphi_{2,t} := \frac{9}{10}\hat{\theta}_t - 2\pi\chi_{A_t} = \frac{9}{10}\bar{\theta} - \frac{\pi}{5}\chi_{A_t}. \tag{27}$$

We will show that:

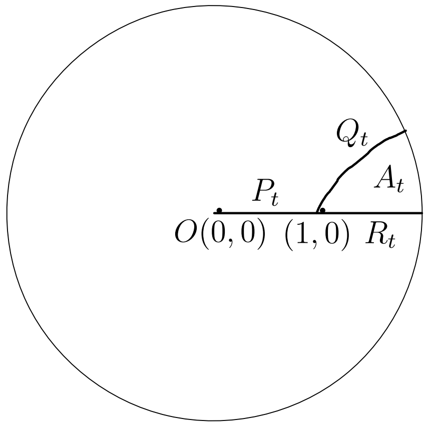

Figure 2: The construction for the general case $p \neq 4$

Lemma 5

(i) For any $t \in (-\frac{1}{4}, 0)$, $\varphi_{1,t}$ is the unique optimal lifting of U_t (up to $2\pi\mathbb{Z}$ additive constants);

(ii) For any $t \in (0, \frac{1}{4})$, $\varphi_{2,t}$ is the unique optimal lifting of U_t (up to $2\pi\mathbb{Z}$ additive constants).

The conclusion of Theorem 1 (in the case of the disc) will then follow from the next result:

Lemma 6

(i) For every $0 < p < 4$ there exists a positive number $\rho_p \in (0, \frac{1}{4})$ such that for any $t \in (-\rho_p, 0)$ we have that $F_0^{(U_t,p)}(\varphi_{1,t}) > F_0^{(U_t,p)}(\varphi_{2,t})$, i.e., the optimal lifting $\varphi_{1,t}$ of U_t is not a minimizer of $F_0^{(U_t,p)}$. Moreover, $\varphi_{2,t}$ is the unique minimizer of $F_0^{(U_t,p)}$ (up to a $2\pi\mathbb{Z}$ translation), for every $t \in (-\rho_p, \rho_p)$.

(ii) For any $p > 4$ there exists $\rho_p \in (0, \frac{1}{4})$ such that $F_0^{(U_t,p)}(\varphi_{2,t}) > F_0^{(U_t,p)}(\varphi_{1,t})$, for each $t \in (0, \rho_p)$, i.e., the optimal lifting $\varphi_{2,t}$ of U_t is not a minimizer of $F_0^{(U_t,p)}$. Moreover, $\varphi_{1,t}$ is the unique minimizer of $F_0^{(U_t,p)}$ (up to a $2\pi\mathbb{Z}$ translation), for every $t \in (-\rho_p, \rho_p)$.

Before proving the above Lemmas, we shall introduce some notations (see Figure 2). Set

$$P_t := \{z \in \mathbb{C} : z = r, \, r \in (0,1)\} \quad \text{and} \quad Q_t := \{z \in \mathbb{C} : z = re^{i(3/4+t)\ln r}, \, r \in (1,2)\}. \tag{28}$$

Then the jump set of U_t is given by

$$S(U_t) = P_t \cup Q_t \cup \{(0,0),(1,0)\}; \tag{29}$$

moreover, we have that

$$\mathcal{H}^1(P_t) = 1 \quad \text{and} \quad \mathcal{H}^1(Q_t) = \sqrt{1+(3/4+t)^2}. \tag{30}$$

We choose the orientation of the jump set $S(U_t)$ to be given by the unit normal vector $\nu_{U_t} \in S^1$ defined by

$$\nu_{U_t}(z) = \begin{cases} (0,1) & z \in P_t, \\ \dfrac{1}{|\gamma_t'(|z|)|}\big(-\gamma_{t,2}'(|z|), \gamma_{t,1}'(|z|)\big) & z \in Q_t, \end{cases}$$

where $\gamma_t(r) = \gamma_{t,1}(r) + i\gamma_{t,2}(r) := re^{i(3/4+t)\ln r}$. Then for any $z \in S(U_t)$ we consider the traces

$$U_t^+(z) = e^{i\frac{9}{10}\bar\theta(z)} \quad \text{and} \quad U_t^-(z) = e^{i\frac{9}{10}(\bar\theta(z)+2\pi)} = e^{i\left(\frac{9}{10}\bar\theta(z)-\frac{\pi}{5}\right)}.$$

We start by giving a useful characterization of a general lifting $\varphi \in BV(\Omega,\mathbb{R})$ of U_t. We can choose the orientation of $S(\varphi)$ to coincide with the orientation of $S(U_t)$ on $S(\varphi)\cap S(U_t)$. Then, we have

$$\varphi^+(z) - \varphi^-(z) = \frac{\pi}{5} + 2\pi n(z), \forall z \in S(U_t) \quad \text{and} \quad \varphi^+(z)-\varphi^-(z) = 2\pi n(z), \forall z \in S(\varphi)\setminus S(U_t),$$

where $n : S(\varphi) \to \mathbb{Z}$ is an integrable function. We define the sets

$$L_\varphi := \{z \in S(\varphi) : \ n(z) \neq 0\} \quad \text{and} \quad L_\varphi^r := \{r \in (0,2) : \ \exists\,\theta \in \mathbb{R}, \ re^{i\theta} \in L_\varphi\}. \tag{31}$$

We next prove the following property:

Lemma 7 *For any lifting $\varphi \in BV(\Omega,\mathbb{R})$ of U_t, we have $\mathcal{H}^1(L_\varphi^r) = 2$.*

Proof. By contradiction, assume that $\mathcal{H}^1(L_\varphi^r) < 2$. Then, there exists a compact set $K \subset (0,2)$ such that $\mathcal{H}^1(K) > 0$ and $L_\varphi^r \cap K = \emptyset$. Consider a sequence of open sets $V_k \subset\subset (0,2)$ such that $K \subset V_k \subset\subset (0,2)$ and $\bigcap_{k=1}^\infty V_k = K$. Now take a sequence of functions $\sigma_k \in C_c^1\big((0,2),\mathbb{R}\big)$ that satisfy $0 \le \sigma_k \le 1$, $\sigma_k(r) = 1$ for any $r \in K$ and $\sigma_k(r) = 0$ for any $r \in (0,2)\setminus V_k$. Define the functions $\delta_k \in C_c^2(\Omega,\mathbb{R})$ by

$$\delta_k(z) := \int_{|z|}^2 \sigma_k(t)dt.$$

For $z = (x,y)$, we denote $\nabla^\perp \delta_k := (-\partial_y\delta_k, \partial_x\delta_k)$. Then we have

$$\int_\Omega \nabla^\perp\delta_k(z)\, d[D\varphi](z) = 0. \tag{32}$$

Since $U_t = e^{i\varphi}$, we obtain from the chain rule (7),

$$D\varphi = D^a\varphi + D^j\varphi = \frac{9}{10}D^a\bar\theta + \frac{\pi}{5}\nu_{U_t}\,\mathcal{H}^1 \llcorner S(U_t) + 2\pi n(\cdot)\nu_\varphi\,\mathcal{H}^1 \llcorner L_\varphi.$$

Therefore, by (32) we infer

$$-2\pi\delta_k(0) + 2\pi\int_{L_\varphi} n(z)\nabla^\perp\delta_k(z)\cdot\nu_\varphi(z)\,d\mathcal{H}^1(z) = 0. \tag{33}$$

Define the sets $W_k := \{z \in \Omega : |z| \in V_k \setminus K\}$, $\forall k \geq 1$. Then by the construction of δ_k, we deduce from (33),

$$\delta_k(0) = \int_{L_\varphi \cap W_k} n(z) \nabla^\perp \delta_k(z) \cdot \nu_\varphi(z)\, d\mathcal{H}^1(z).$$

Since $|\nabla^\perp \delta_k| \leq 1$, it follows that

$$|\delta_k(0)| \leq \int_{L_\varphi \cap W_k} |n(z)|\, d\mathcal{H}^1(z) \leq \frac{1}{\pi} \int_{L_\varphi \cap W_k} |\varphi^+(z) - \varphi^-(z)|\, d\mathcal{H}^1(z) \leq \frac{1}{\pi} \int_{W_k} |D\varphi|.$$

Using $\cap_{k=1}^\infty W_k = \emptyset$, we get that

$$\lim_{k \to \infty} \delta_k(0) = 0. \tag{34}$$

On the other hand, according to the definition of δ_k, we have

$$\delta_k(0) = \int_0^2 \sigma_k(t) dt \geq \int_K 1\, dt = \mathcal{H}^1(K) > 0,$$

which leads to a contradiction to (34). This completes the proof of Lemma 7. $\square$

We now present the proofs of Lemmas 5 and 6:

Proof of Lemma 5. The jump set of $\varphi_{1,t}$ and $\varphi_{2,t}$ are

$$S(\varphi_{1,t}) = S(U_t) = P_t \cup Q_t \cup \{(0,0),(1,0)\} \quad \text{and} \quad S(\varphi_{2,t}) = P_t \cup Q_t \cup R_t \cup \{(0,0),(1,0)\}, \tag{35}$$

where $R_t := \{z \in \mathbb{C} : z = r,\ r \in (1,2)\}$. Moreover, the size of the jump is

$$|\varphi_{1,t}^+(z) - \varphi_{1,t}^-(z)| = \frac{9\pi}{5}, \quad \forall z \in P_t \cup Q_t$$

and

$$|\varphi_{2,t}^+(z) - \varphi_{2,t}^-(z)| = \begin{cases} \frac{9\pi}{5} & \text{if} \quad z \in P_t, \\ \frac{\pi}{5} & \text{if} \quad z \in Q_t, \\ 2\pi & \text{if} \quad z \in R_t. \end{cases}$$

Therefore, by (30), it follows that

$$\begin{aligned} \int_\Omega |D^j \varphi_{1,t}| &= \frac{9\pi}{5} + \frac{9\pi}{5}\sqrt{1 + (3/4+t)^2}; \\ \int_\Omega |D^j \varphi_{2,t}| &= \frac{9\pi}{5} + \frac{\pi}{5}\sqrt{1 + (3/4+t)^2} + 2\pi. \end{aligned} \tag{36}$$

Hence, we have

$$\begin{aligned} &\int_\Omega |D^j \varphi_{1,t}| < \int_\Omega |D^j \varphi_{2,t}|, \quad \forall t \in (-1/4, 0), \\ &\int_\Omega |D^j \varphi_{1,t}| > \int_\Omega |D^j \varphi_{2,t}|, \quad \forall t \in (0, 1/4), \\ &\int_\Omega |D^j \varphi_{1,0}| = \int_\Omega |D^j \varphi_{2,0}|. \end{aligned} \tag{37}$$

Let now $\varphi \in BV(\Omega, \mathbb{R})$ be an arbitrary lifting of U_t. From (11) it follows that $\int_\Omega |D^a\varphi| = \int_\Omega |D^a U_t|$ and $\int_\Omega |D^c\varphi| = \int_\Omega |D^c U_t| = 0$. We choose an orientation of $S(\varphi)$ that coincides with the orientation of $S(U_t)$ on $S(\varphi) \cap S(U_t)$. Put

$$\begin{cases} x_\varphi := \mathcal{H}^1(L_\varphi \cap P_t), \quad y_\varphi := \mathcal{H}^1(L_\varphi \cap Q_t), \\ w_\varphi := \mathcal{H}^1(S(\varphi) \setminus S(U_t)) = \mathcal{H}^1\big(L_\varphi \setminus (P_t \cup Q_t)\big), \\ z_\varphi := w_\varphi + x_\varphi + \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}}, \end{cases} \tag{38}$$

where P_t and Q_t are defined in (28) and L_φ is given in (31). Consider the following decomposition of L^r_φ (defined in (31)):

$$L^r_\varphi = A^r_\varphi \cup B^r_\varphi \cup D^r_\varphi \quad \text{ a.e. in } (0,2),$$

where

$$\begin{cases} A^r_\varphi := \{r \in (0,1) : \exists\, \theta \in \mathbb{R},\ re^{i\theta} \in L_\varphi \cap P_t\}, \\ B^r_\varphi := \{r \in (1,2) : \exists\, \theta \in \mathbb{R},\ re^{i\theta} \in L_\varphi \cap Q_t\}, \\ D^r_\varphi := \{r \in (0,2) : \exists\, \theta \in \mathbb{R},\ re^{i\theta} \in L_\varphi \setminus (P_t \cup Q_t)\}. \end{cases} \tag{39}$$

Note that $A^r_\varphi \cap B^r_\varphi = \emptyset$, but A^r_φ (resp. B^r_φ) and D^r_φ are not necessarily disjoint. We have

$$\mathcal{H}^1(A^r_\varphi) = x_\varphi \quad \text{and} \quad \mathcal{H}^1(B^r_\varphi) = \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}},$$

where the last equality follows by the construction of Q_t. It is clear then that

$$w_\varphi \geq \mathcal{H}^1\big(D^r_\varphi\big) \geq \mathcal{H}^1\big(L^r_\varphi \setminus (A^r_\varphi \cup B^r_\varphi)\big) = \mathcal{H}^1\big(L^r_\varphi\big) - x_\varphi - \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}}.$$

By Lemma 7 we have $\mathcal{H}^1(L^r_\varphi) = 2$. Therefore,

$$w_\varphi \geq 2 - x_\varphi - \frac{y_\varphi}{\sqrt{1+(3/4+t)^2}}, \quad \text{i.e.,} \quad z_\varphi \geq 2. \tag{40}$$

By (30), we deduce that

$$(x_\varphi, y_\varphi, z_\varphi) \in M_t := \{(x,y,z) \in \mathbb{R}^3 :\ 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1+(3/4+t)^2},\ z \geq 2\}. \tag{41}$$

We define the function $\Phi_t : M_t \to \mathbb{R}$ by

$$\Phi_t(x,y,z) := 2\pi z - \frac{2\pi}{5}x + \frac{2\pi\big(4\sqrt{1+(3/4+t)^2} - 5\big)}{5\sqrt{1+(3/4+t)^2}}y + \frac{\pi}{5}\Big(1 + \sqrt{1+(3/4+t)^2}\Big).$$

It is easy to check that for $t > 0$ the unique minimum point of Φ_t on the set M_t is achieved at the point $(1,0,2)$. Similarly, if $t < 0$ then Φ_t attains its unique minimum on the set M_t at $(x,y,z) = \big(1, \sqrt{1+(3/4+t)^2}, 2\big)$.

On the other hand, from (29) we infer

$$\begin{aligned} \int_\Omega |D^j\varphi| &\geq \int_{S(\varphi)\setminus S(U_t)} |\varphi^+ - \varphi^-| + \int_{(L_\varphi \cap P_t)\cup(L_\varphi \cap Q_t)} |\varphi^+ - \varphi^-| + \int_{(P_t \cup Q_t)\setminus L_\varphi} |\varphi^+ - \varphi^-| \\ &\geq 2\pi w_\varphi + \Big(2\pi - \frac{\pi}{5}\Big)(x_\varphi + y_\varphi) + \frac{\pi}{5}\Big(1 + \sqrt{1+(3/4+t)^2} - x_\varphi - y_\varphi\Big) \\ &= \Phi_t(x_\varphi, y_\varphi, z_\varphi). \end{aligned} \tag{42}$$

Therefore,

$$
\begin{aligned}
&\int_\Omega |D^j\varphi| \geq \Phi_t(x_\varphi, y_\varphi, z_\varphi) \geq \Phi_t\big(1, \sqrt{1+(3/4+t)^2}, 2\big) = \int_\Omega |D^j\varphi_{1,t}|, \quad \text{if } t \in (-1/4, 0),\\
&\int_\Omega |D^j\varphi| \geq \Phi_t(x_\varphi, y_\varphi, z_\varphi) \geq \Phi_t(1, 0, 2) = \int_\Omega |D^j\varphi_{2,t}|, \quad \text{if } t \in (0, 1/4).
\end{aligned}
\tag{43}
$$

We conclude that for $t \in (-1/4, 0)$, $\varphi_{1,t}$ is an optimal lifting of U_t while for $t \in (0, 1/4)$, $\varphi_{2,t}$ is an optimal lifting of U_t.

It remains to prove the uniqueness of the optimal lifting of U_t. Let φ be an arbitrary optimal lifting of U_t. Then all inequalities in (42) and (43) become equalities.
(i) In the case of $t \in (-1/4, 0)$, we deduce that $x_\varphi = 1$, $y_\varphi = \sqrt{1+(3/4+t)^2}$, $w_\varphi = 0$ (hence, $S(\varphi) = S(U_t)$). Moreover, by (42),

$$
|\varphi^+ - \varphi^-| = \frac{9\pi}{5} \quad \mathcal{H}^1\text{-a.e. in } S(\varphi).
$$

Since every lifting has the same diffuse part (see (11)), it follows that

$$
D(\varphi - \varphi_{1,t}) = 0 \quad \text{in } \Omega.
$$

Since Ω is connected, we conclude that $\varphi - \varphi_{1,t}$ is constant in Ω.
(ii) In the case $t \in (0, 1/4)$ we obtain $x_\varphi = 1$, $y_\varphi = 0$, $w_\varphi = 1$. Moreover, by (42),

$$
|\varphi^+ - \varphi^-| = \begin{cases} \frac{9\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi) \cap P_t,\\ \frac{\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi) \cap Q_t,\\ 2\pi & \mathcal{H}^1\text{-a.e. in } S(\varphi) \setminus (P_t \cup Q_t). \end{cases}
$$

Then, according to (11), it follows that

$$
D(\varphi - \varphi_{2,t}) = 2\pi\Big(\nu_{\varphi_{2,t}} \mathcal{H}^1 \llcorner R_t - \nu_\varphi \mathcal{H}^1 \llcorner \big(S(\varphi) \setminus S(U_t)\big)\Big).
$$

We deduce that for every function $\delta \in C_c^1(\Omega)$,

$$
\int_{S(\varphi)\setminus S(U_t)} \frac{\partial \delta}{\partial \tau_\varphi}\, d\mathcal{H}^1 = \int_{S(\varphi)\setminus S(U_t)} \nabla^\perp \delta \cdot \nu_\varphi \, d\mathcal{H}^1 = \delta(1, 0),
$$

where τ_φ stands for the tangent vector to the $\mathcal{H}^1$-rectifiable set $S(\varphi) \setminus S(U_t)$. Using the same technique as in [7], since $\mathcal{H}^1\big(S(\varphi) \setminus S(U_t)\big) = \operatorname{dist}((0,1), \partial\Omega) = 1$, we conclude that $S(\varphi) \setminus S(U_t)$ coincides with R_t (which is the geodesic line between the point $(0, 1)$ and $\partial\Omega$). Thus, $D(\varphi - \varphi_{2,t}) = 0$ in Ω, i.e., $\varphi - \varphi_{2,t}$ is constant in Ω. This completes the proof of Lemma 5. $\square$

Proof of Lemma 6. Let $p > 0$. By Lemma 1 we compute

$$
\begin{aligned}
F_0^{(U_t,p)}(\varphi_{1,t}) &= \big(1 + \sqrt{1+(3/4+t)^2}\big) \int_{-9\pi/10}^{9\pi/10} 2|e^{is} - 1|^{p/2} ds\\
&= 2^{p/2+3}\big(1 + \sqrt{1+(3/4+t)^2}\big) \int_0^{9\pi/20} \sin^{p/2} s\, ds\\
&= 2^{p/2+3} \int_0^{9\pi/20} \sin^{p/2} s\, ds + 2^{p/2+3}\sqrt{1+(3/4+t)^2} \int_{\pi/20}^{\pi/2} \cos^{p/2} s\, ds.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
F_0^{(U_t,p)}(\varphi_{2,t}) &= \int_0^{9\pi/10} 4|e^{is}-1|^{p/2} ds + \sqrt{1+(3/4+t)^2}\int_0^{\pi/10} 4|e^{is}-1|^{p/2} ds \\
&\quad + \int_0^{\pi} 4|e^{is}-1|^{p/2} ds \\
&= 2^{p/2+3}\Big(\int\limits_0^{9\pi/20} \sin^{p/2} s\, ds + \sqrt{1+(3/4+t)^2} \int\limits_0^{\pi/20} \sin^{p/2} s\, ds + \int\limits_0^{\pi/2} \cos^{p/2} s\, ds \Big).
\end{aligned}
$$

Therefore, we infer that

$$
\begin{aligned}
&2^{-p/2-3}\big(F_0^{(U_t,p)}(\varphi_{1,t}) - F_0^{(U_t,p)}(\varphi_{2,t})\big) = \\
&= \big(\sqrt{1+(3/4+t)^2}-1\big) \int\limits_0^{\pi/2} \cos^{p/2} s\, ds - \sqrt{1+(3/4+t)^2} \int\limits_0^{\pi/20} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds \\
&= \big(\sqrt{1+(3/4+t)^2}-1\big) \int\limits_0^{\pi/4} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds - \sqrt{1+(3/4+t)^2} \int\limits_0^{\pi/20} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds \\
&= \frac{1}{5}\int_0^{\pi/4} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds \cdot \Big(5\big(\sqrt{1+(3/4+t)^2}-1\big) - c_p\sqrt{1+(3/4+t)^2}\Big),
\end{aligned}
\tag{44}
$$

where we denoted

$$
c_p := \frac{5\int_0^{\pi/20} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds}{\int_0^{\pi/4} \big(\cos^{p/2} s + \sin^{p/2} s\big) ds} \in (0,5).
$$

Since the function

$$
s \in (0, \frac{\pi}{4}) \mapsto \big(\cos^{p/2} s + \sin^{p/2} s\big)
$$

is increasing for $0 < p < 4$ and decreasing for $p > 4$, it turns out that

$$
c_p < 1, \quad \forall p \in (0,4) \qquad \text{and} \qquad c_p > 1, \quad \forall p \in (4,\infty).
$$

Therefore, by (44), for any $p \in (0,4)$ there exists $0 < \rho_p < 1/4$ such that

$$
F_0^{(U_t,p)}(\varphi_{1,t}) > F_0^{(U_t,p)}(\varphi_{2,t}) \qquad \forall t \in (-\rho_p, \rho_p). \tag{45}
$$

Similarly, for any $p \in (4,\infty)$, there exists $0 < \rho_p < 1/4$ such that

$$
F_0^{(U_t,p)}(\varphi_{1,t}) < F_0^{(U_t,p)}(\varphi_{2,t}) \qquad \forall t \in (-\rho_p, \rho_p). \tag{46}
$$

Now we prove that for any $t \in (-\rho_p, \rho_p)$, $\varphi_{2,t}$ (resp. $\varphi_{1,t}$) is the unique minimizer of $F_0^{(U_t,p)}$ if $p \in (0,4)$ (resp. $p > 4$). Let $\varphi \in BV(\Omega, \mathbb{R})$ be an arbitrary lifting of U_t. We choose an orientation on $S(\varphi)$ that coincides with the orientation of $S(U_t)$ on $S(\varphi) \cap S(U_t)$. In the following we use the same notations as in the proof of Lemma 5 (see (38), (39) and (41)). We define the function

$\Psi_t : M_t \to \mathbb{R}$ by

$$\begin{aligned}
\Psi_t(x,y,z) :&= f^{(p)}(2\pi)z - \Big(f^{(p)}(2\pi) + f^{(p)}\big(\frac{\pi}{5}\big) - f^{(p)}\big(\frac{9\pi}{5}\big)\Big)x \\
&\quad + \Big(f^{(p)}\big(\frac{9\pi}{5}\big) - \frac{f^{(p)}(2\pi)}{\sqrt{1+(3/4+t)^2}} - f^{(p)}\big(\frac{\pi}{5}\big)\Big)y + f^{(p)}\big(\frac{\pi}{5}\big)\Big(1+\sqrt{1+(3/4+t)^2}\Big) \\
&= f^{(p)}(2\pi)z - \Big(f^{(p)}(2\pi) + f^{(p)}\big(\frac{\pi}{5}\big) - f^{(p)}\big(\frac{9\pi}{5}\big)\Big)x \\
&\quad + \frac{y}{\sqrt{1+(3/4+t)^2}}\Big(F_0^{(U_t,p)}(\varphi_{1,t}) - F_0^{(U_t,p)}(\varphi_{2,t})\Big) + f^{(p)}\big(\frac{\pi}{5}\big)\Big(1+\sqrt{1+(3/4+t)^2}\Big).
\end{aligned}$$

By (45) and (46), it can be easily checked that: if $p \in (0,4)$ and $t \in (-\rho_p, \rho_p)$ then the unique minimal point of Ψ_t in the set M_t is achieved in $(1,0,2)$, while if $p > 4$ and $t \in (-\rho_p, \rho_p)$ then Ψ_t has also a unique minimal point in M_t for $(x,y,z) = \big(1, \sqrt{1+(3/4+t)^2}, 2\big)$. Using the same argument as in the proof of Lemma 5, it follows that

$$\begin{aligned}
\frac{F_0^{(U_t,p)}(\varphi)}{2} &\geq \int_{S(\varphi)\setminus S(U_t)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^1 + \int_{(L_\varphi\cap P_t)\cup(L_\varphi\cap Q_t)} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^1 \\
&\quad + \int_{(P_t\cup Q_t)\setminus L_\varphi} f^{(p)}(|\varphi^+ - \varphi^-|)\, d\mathcal{H}^1 \\
&\geq f^{(p)}(2\pi)w_\varphi + f^{(p)}\Big(2\pi - \frac{\pi}{5}\Big)(x_\varphi + y_\varphi) + f^{(p)}\Big(\frac{\pi}{5}\Big)\Big(1+\sqrt{1+(3/4+t)^2} - x_\varphi - y_\varphi\Big) \\
&= \Psi_t(x_\varphi, y_\varphi, z_\varphi).
\end{aligned} \tag{47}$$

Therefore, for every $t \in (-\rho_p, \rho_p)$,

$$\begin{cases}
F_0^{(U_t,p)}(\varphi) \geq 2\Psi_t(x_\varphi, y_\varphi, z_\varphi) \geq 2\Psi_t\big(1, \sqrt{1+(3/4+t)^2}, 2\big) = F_0^{(U_t,p)}(\varphi_{1,t}) & \text{if } p > 4, \\
F_0^{(U_t,p)}(\varphi) \geq 2\Psi_t(x_\varphi, y_\varphi, z_\varphi) \geq 2\Psi_t(1,0,2) = F_0^{(U_t,p)}(\varphi_{2,t}) & \text{if } p \in (0,4).
\end{cases} \tag{48}$$

It follows that for any $t \in (-\rho_p, \rho_p)$, $\varphi_{1,t}$ is a minimizer of $F_0^{(U_t,p)}$ if $p > 4$, and $\varphi_{2,t}$ is a minimizer of $F_0^{(U_t,p)}$ if $p \in (0,4)$. It remains to prove the uniqueness of the minimizer of $F_0^{(U_t,p)}$ for any $t \in (-\rho_p, \rho_p)$. Let φ be a lifting of U_t that minimizes the energy $F_0^{(U_t,p)}$. Then all inequalities in (47) and (48) become equalities. Next we distinguish two cases:

(i) In the case of $p > 4$ we deduce that $x_\varphi = 1$, $y_\varphi = \sqrt{1+(3/4+t)^2}$, $w_\varphi = 0$ (hence, $S(\varphi) = S(U_t)$). Moreover, by Lemma 1 and (47),

$$|\varphi^+ - \varphi^-| = \frac{9\pi}{5} \quad \mathcal{H}^1\text{-a.e. in } S(\varphi).$$

Since every lifting has the same diffuse part (see (11)), it follows that

$$D(\varphi - \varphi_{1,t}) = 0 \quad \text{in } \Omega.$$

Since Ω is connected, we conclude that $\varphi - \varphi_{1,t}$ is constant in Ω.

(ii) In the case $p \in (0,4)$ we obtain that $x_\varphi = 1$, $y_\varphi = 0$, $w_\varphi = 1$. Moreover, by (47)

$$|\varphi^+ - \varphi^-| = \begin{cases}
\frac{9\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi)\cap P_t, \\
\frac{\pi}{5} & \mathcal{H}^1\text{-a.e. in } S(\varphi)\cap Q_t, \\
2\pi & \mathcal{H}^1\text{-a.e. in } S(\varphi)\setminus (P_t\cup Q_t).
\end{cases}$$

Then, by the same argument as in the end of the proof of Lemma 5, we conclude that $\varphi - \varphi_{2,t}$ is constant in Ω. $\square$

In the following, we shall adapt our construction of the family $\mathcal{U}$ to the general case of an arbitrary domain G:

Proof of *(ii)* in Theorem 1. Assume that G is an arbitrary bounded domain in $\mathbb{R}^N$ for $N \geq 2$. We construct a family of functions $\tilde{\mathcal{U}} = \{\tilde{U}_t\}_{t\in(-1/4,1/4)}$ in $BV(G, S^1)$ that will have the same behavior as the family $\mathcal{U} = \{U_t\}_{t\in(-1/4,1/4)}$, defined in (26) over the set $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 4\}$. Let us introduce the sets

$$\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 16\},$$

$$G_1 := \Omega \times (-1/2, 1/2)^{N-2} \subset \mathbb{R}^N \quad \text{and} \quad G_2 := \Omega_1 \times (-1, 1)^{N-2} \subset \mathbb{R}^N.$$

For $t \in (-1/4, 1/4)$, set also

$$H_t := \{\, (x_1, x_2) \in \Omega_1 : (x_1, x_2) = re^{i\theta},\ r \in (1, 4),\ 0 < \theta < (3/4 + t)\ln r \,\},$$

and define $\tilde{H}_t := H_t \times (-1, 1)^{N-2} \subset \mathbb{R}^N$. As before, by translating and shrinking homotopically the set G_2, we may suppose that $G_2 \subset G$. We write $x = (x_1, x_2, \ldots, x_N) = (x_1, x_2, x') \in \mathbb{R}^N$. Next we define the function $\tilde{U}_t \in BV(G, S^1)$ by

$$\tilde{U}_t(x) := \begin{cases} U_t(x_1, x_2) & x \in G_1, \\ 1 & x \in \tilde{H}_t \setminus G_1, \\ -1 & \text{otherwise.} \end{cases} \tag{49}$$

Recall the liftings $\varphi_{1,t}, \varphi_{2,t} \in BV(\Omega, \mathbb{R})$ of U_t defined in (27). Then, consider the liftings $\Phi_{1,t}, \Phi_{2,t} \in BV(G, \mathbb{R})$ of $\tilde{U}_t$ given by

$$\Phi_{1,t}(x) := \begin{cases} \varphi_{1,t}(x_1, x_2) & x \in G_1, \\ 2\pi & x \in \tilde{H}_t \setminus G_1, \\ \pi & \text{otherwise} \end{cases} \quad \text{and} \quad \Phi_{2,t}(x) := \begin{cases} \varphi_{2,t}(x_1, x_2) & x \in G_1, \\ 0 & x \in \tilde{H}_t \setminus G_1, \\ \pi & \text{otherwise.} \end{cases} \tag{50}$$

The jump part of these liftings enjoys the following property: for every $j = 1, 2$, and every $t \in (-1/4, 1/4)$ we have

$$S(\Phi_{j,t}) \setminus G_1 = S(\tilde{U}_t) \setminus G_1 \ \text{ and } \ \left|\Phi_{j,t}^+(x) - \Phi_{j,t}^-(x)\right| = d_{S^1}\big(\tilde{U}_t^+(x), \tilde{U}_t^-(x)\big)\ \mathcal{H}^{N-1}\text{-a.e. in } S(\Phi_{j,t}) \setminus G_1. \tag{51}$$

In the sequel we will prove that the analog results to those of Lemmas 5 and 6 hold for the functions $\Phi_{j,t}$, $j = 1, 2$.

Step 1. For $j = 1, 2$, $\Phi_{j,t}$ is the unique optimal lifting of $\tilde{U}_t$ (up to $2\pi\mathbb{Z}$ constants) if t is between 0 and $(-1)^j/4$.

Indeed, let $\Phi : G \to \mathbb{R}$ be an arbitrary lifting of $\tilde{U}_t$ on G. First notice that by (12), we have that

$$\int_{G\setminus G_1} |D^a\Phi| + \int_{G\setminus G_1} |D^c\Phi| = \int_{G\setminus G_1} |D^a\tilde{U}_t| + \int_{G\setminus G_1} |D^c\tilde{U}_t| = 0.$$

Using Lemma 5 it follows that

$$\begin{aligned}\int_G |D\Phi| &= \int_{G\setminus G_1} |D\Phi| + \int_{G_1} |D\Phi| \\ &= \int_{S(\Phi)\setminus G_1} |\Phi^+ - \Phi^-|\, d\mathcal{H}^{N-1} + \int_{G_1} |D\Phi| \\ &\geq \int_{S(\tilde{U}_t)\setminus G_1} d_{S^1}(\tilde{U}_t^+, \tilde{U}_t^-) d\mathcal{H}^{N-1} + \int\limits_{(-1/2,1/2)^{N-2}} dx' \int\limits_{\Omega\times\{x'\}} \Big|\big(\frac{\partial\Phi}{\partial x_1}, \frac{\partial\Phi}{\partial x_2}\big)\Big| \\ &\geq \int_{S(\tilde{U}_t)\setminus G_1} d_{S^1}(\tilde{U}_t^+, \tilde{U}_t^-) d\mathcal{H}^{N-1} + \int\limits_{\Omega} |D\varphi_{j,t}| = \int_G |D\Phi_{j,t}|, \end{aligned} \tag{52}$$

i.e., $\Phi_{j,t}$ is an optimal lifting of $\tilde{U}_t$ if t is between 0 and $(-1)^j/4$. It remains to show the uniqueness of the optimal lifting. For that, let Φ be an arbitrary optimal lifting of $\tilde{U}_t$. Then we must have equalities in (52) and therefore we obtain:

$$S(\Phi)\setminus G_1 = S(\tilde{U}_t)\setminus G_1 \quad \text{and} \quad \big|\Phi^+(x) - \Phi^-(x)\big| = d_{S^1}\big(\tilde{U}_t^+(x), \tilde{U}_t^-(x)\big) \quad \mathcal{H}^{N-1}\text{-a.e. in } S(\Phi_{j,t})\setminus G_1\,, \tag{53}$$

and for almost every $x' \in (-1/2, 1/2)^{N-2}$, the restriction of Φ to $\Omega \times \{x'\}$ is an optimal lifting of U_t. Therefore, the jump set of Φ satisfies:

$$S(\Phi) \cap G_1 = S(\varphi_{j,t}) \times (-1/2, 1/2)^{N-2} = S(\Phi_{j,t}) \cap G_1.$$

By (11), it follows that $D(\Phi - \Phi_{j,t}) = 0$ in $G_1 \setminus S(\Phi_{j,t})$, i.e., $\Phi - \Phi_{j,t}$ is constant on all j connected components of $G_1 \setminus S(\Phi_{j,t})$, $j = 1, 2$. The optimality of Φ does not allow any jumps for $\Phi - \Phi_{j,t}$ on $S(\Phi_{j,t}) \cap G_1$. Hence, by (53), we conclude that $\Phi - \Phi_{j,t}$ is constant in G.

Step 2. For every $p \in (4, \infty)$ (resp. $p \in (0, 4)$), there exists $\rho_p \in (0, \frac{1}{4})$ such that for any $0 < t < \rho_p$ (resp. $-\rho_p < t < 0$), we have

$$F_0^{(\tilde{U}_t,p)}(\Phi_{2,t}) > F_0^{(\tilde{U}_t,p)}(\Phi_{1,t}) \quad (\text{resp. } F_0^{(\tilde{U}_t,p)}(\Phi_{1,t}) > F_0^{(\tilde{U}_t,p)}(\Phi_{2,t})\ ),$$

i.e., the optimal lifting of $\tilde{U}_t$ is not a minimizer of $F_0^{(\tilde{U}_t,p)}$ for the above ranges of p and t.

Indeed, let us prove the claim for $p > 4$ (the other case follows using the same argument). Take $\rho_p \in (0, 1/4)$ as given by Lemma 6. Then, by Step 1 and Lemma 6, we deduce that for $t \in (0, \rho_p)$,

$$\begin{aligned} F_0^{(\tilde{U}_t,p)}(\Phi_{2,t}) &= \int_{S(\Phi_{2,t})\setminus G_1} f^{(p)}(|\Phi_{2,t}^+ - \Phi_{2,t}^-|)\, d\mathcal{H}^{N-1} + \int_{G_1\cap S(\Phi_{2,t})} f^{(p)}(|\Phi_{2,t}^+ - \Phi_{2,t}^-|) d\mathcal{H}^{N-1} \\ &= \int_{S(\tilde{U}_t)\setminus G_1} f^{(p)}\big(d_{S^1}(\tilde{U}_t^+, \tilde{U}_t^-)\big) d\mathcal{H}^{N-1} + \int_{\Omega\cap S(\varphi_{2,t})} f^{(p)}(|\varphi_{2,t}^+ - \varphi_{2,t}^-|) d\mathcal{H}^1 \\ &> \int_{S(\tilde{U}_t)\setminus G_1} f^{(p)}\big(d_{S^1}(\tilde{U}_t^+, \tilde{U}_t^-)\big) d\mathcal{H}^{N-1} + \int_{\Omega\cap S(\varphi_{1,t})} f^{(p)}(|\varphi_{1,t}^+ - \varphi_{1,t}^-|)\, d\mathcal{H}^1 \\ &= F_0^{(\tilde{U}_t,p)}(\Phi_{1,t}). \end{aligned}$$

As before, one can also obtain that for any $t \in (-\rho_p, \rho_p)$, $\Phi_{2,t}$ (resp. $\Phi_{1,t}$) is the unique minimizer of $F_0^{(\tilde{U}_t,p)}$ if $p \in (0, 4)$ (resp. $p > 4$). □

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